Epsilon: Jurnal Matematika Murni dan Terapan
http://ppjp.ulm.ac.id/journals/index.php/epsilon
p-ISSN: 1978-4422 | e-ISSN 2656-7660
Vol.19 No.2 (Desember 2025) Hal. 156-165

LOWER LEVEL SUBGRUPOID

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ABSTRACT

This paper investigates the structure of anti-fuzzy subgroupoids within the framework of groupoids, aiming to extend fuzzy subgroup theory beyond traditional group-based algebraic systems. An anti-fuzzy subgroupoid is a fuzzy subset in which the degree of membership of the product of two elements is less than or equal to the maximum of their membership degrees. This reflects a form of incompatibility or non-membership, offering a complementary perspective to fuzzy subgroupoids. Although fuzzy approaches have been widely applied to groups and semigroups, the study of groupoid structures—algebraic systems that do not require identity or inverse elements—remains limited, particularly in the context of anti-fuzzy theory. This work formally characterizes anti-fuzzy subgroupoids and analyzes their relationship with lower-level subsets. It is shown that every subgroupoid can be represented as a lower-level subset of a suitably constructed anti-fuzzy subgroupoid, and that two lower-level subsets are equal if and only if no element exists with a strict membership value between the corresponding thresholds. Using a deductive and axiomatic methodology, this study contributes to the theoretical development of fuzzy structures in non-classical algebra. It provides a foundation for future applications in uncertainty-based decision systems.

Keywords: groupoid, subgroupoid, lower-level subset, anti-fuzzy subgroupoid

Received: 13 May 2025, Accepted: 18 July 2025, Published: 27 October 2025

INTRODUCTION

Since the introduction of fuzzy set theory by Zadeh (1965) and its application to group theory by Rosenfeld (1971), the theory of fuzzy subgroups has undergone significant development. Numerous approaches have been proposed to address uncertainty in algebraic structures, including fuzzy subgroups (Thakur et al., 2023), intuitionistic fuzzy subgroups (Divya Mary Daise et al., 2020, 2021), Pythagorean fuzzy subgroups (Bhunia et al., 2021), and Fermatean fuzzy subgroups (Mishra et al., 2023).

These advancements not only vary in fuzzy types but also extend to the underlying algebraic frameworks. For example, soft groups and soft automorphisms have been introduced (Alajlan & Alghamdi, 2023), and further generalizations such as soft groupoids and soft subgroupoids have been explored (Oguz, G., Icen, I. ve Gürsoy, 2020). More recently, Abdurrahman (2024)

contributed to this area by discussing the concept of soft groupoids and their algebraic properties, offering a flexible framework for modeling uncertainty in non-classical algebraic systems.

In addition, harmonized fuzzy structures such as Harmonized Fuzzy Groups have been proposed to unify various fuzzy group variants (Onasanya et al., 2022). Within the context of fuzzy subgroups, several studies have examined level-based structures, including α-level subgroups (Rajeswari, 2019), lower-level subgroups (Gayen et al., 2021), and level sets of t-fuzzy groups (Chon, 2001).

However, most of these studies remain focused on groups or semigroups, with limited exploration of groupoid structures—algebraic systems that do not require identity or inverse elements. Furthermore, although the concept of anti-fuzzy subgroups has been introduced and developed (Ahmad et al., 2021; Gayen et al., 2021; Kalaiarasi et al., 2022), its application within groupoid structures remains largely unexplored.

The research gap lies in the absence of a systematic investigation connecting anti-fuzzy subgroupoid structures with the properties of lower-level subsets in the context of groupoids. This connection is crucial, as lower-level subsets can represent the degree of non-membership or incompatibility of elements within a system, which is highly relevant in uncertainty-based decision-making frameworks.

LITERATURE REVIEW

This section outlines fundamental concepts such as groupoids, fuzzy subsets, fuzzy subgroupoids, and anti-fuzzy subgroupoids, which form the theoretical foundation of this study. Understanding these concepts is essential to support the analysis and proofs presented in the results and discussion sections.

Definition 2.1. (Abdurrahman, 2022, 2024) A non-empty set S is called a groupoid if it is equipped with a binary operation $\star$ such that $a \star c \in S$ for all $a, c \in S$. This condition ensures the closure property of the operation S.

Definition 2.2. (Kandasamy, 2003) Let $(S,\star)$ be a groupoid. A non-empty subset $H \subseteq S$ is called a subgroupoid S if it is closed under the operation $\star$, i.e., $(S,\star)$ itself forms a groupoid.

For simplicity, the operation $\star$ may be denoted by juxtaposition, i.e., ac represents $a \star c$.

Definition 2.3. (Abdurrahman, 2024) Let H be a non-empty subset of a groupoid S. The subset H is called a subgroupoid of S if and only if $ac \in H$ for all $a, c \in H$.

Definition 2.4. A subgroupoid H of a groupoid S is called an ideal of S if and only if for every $s \in S$ and $a \in H$, the conditions $as \in H$ and $sa \in H$.

Definition 2.5. Let S be a non-empty set. A fuzzy subset $\mathfrak{f}$ of S is defined as a function $\mathfrak{f}: S \longrightarrow [0,1]$. The fuzzy subset $\mathfrak{f}$ can be expressed as $\mathfrak{f} = \{(a, \mathfrak{f}(a)) | a \in S\}$, where $\mathfrak{f}(a)$ Represents the degree of membership of x.

Subsequently, Atanassov (Atanassov, 1986) extended Zadeh's concept by introducing a non-membership degree $\mathfrak{d}: S \longrightarrow [0,1]$, which satisfies $0 \leq \mathfrak{f}(a) + \mathfrak{d}(a) \leq 1$ for all $a \in S$, where $\mathfrak{f}$ is the membership function and $\mathfrak{d}$ is the non-membership function**.**

Definition 2.6. For any fuzzy subset $\mathfrak{f}$ of a groupoid S, the complement $\mathfrak{f}^c$ is the fuzzy subset defined by:

$$\mathfrak{f}^c(a) = 1 - \mathfrak{f}(a) \text{ For all } a \in S.$$

Definition 2.7. A fuzzy subset $\mathfrak{f}$ of a groupoid S is termed a fuzzy subgroupoid of S if it satisfies the following condition for all $a, c \in S$:

$$\mathfrak{f}(ac) \geq \mathfrak{f}(a) \wedge \mathfrak{f}(c).$$

In algebraic structures, this means that the function $\mathfrak{f}$ assigns to each element a value in the interval $[0, 1]$ representing how strongly the element belongs to the fuzzy subgroupoid. For example, if S is a groupoid under multiplication and $\mathfrak{f}(a) = 0{,}8$ then element a is considered to have an 80% degree of membership in the fuzzy subgroupoid.

Definition 2.8. Let $\mathfrak{f}$ be a fuzzy subset of groupoid S. For any $\sigma \in [0,1]$, the level subset of $\mathfrak{f}$, denoted by $\mathfrak{f}_\sigma$, is defined as:

$$\mathfrak{f}_\sigma \stackrel{\text{def}}{=} \{a \in S | \mathfrak{f}(a) \geq \sigma\}.$$

Definition 2.9. Let $\mathfrak{f}$ and $\mathfrak{d}$ be a fuzzy subset of a groupoid S. The intersection of $\mathfrak{f}$ and $\mathfrak{d}$, denoted by $\mathfrak{f} \widehat{\cap} \mathfrak{d}$, is defined by their membership functions as:

$$\mathfrak{f} \widehat{\cap} \mathfrak{d}(a) \stackrel{\text{def}}{=} \mathfrak{f}(a) \wedge \mathfrak{d}(a)$$

for all $a \in S$.

Definition 2.10. A fuzzy subset $\mathfrak{f}$ of a groupoid S is an anti-fuzzy subgroupoid if it satisfies:

$$\mathfrak{f}(ac) \leq \mathfrak{f}(a) \vee \mathfrak{f}(c)$$

for all $a, c \in S$.

In algebraic systems where a set function $\mathfrak{f}$ qualifies as an anti-fuzzy subgroupoid of groupoid S, $\mathfrak{f}$ operates as a non-membership indicator, assigning to each element $a \in S$ a numerical value within $[0,1]$ that quantifies its exclusion level from $\mathfrak{f}$. Taking a groupoid S with multiplicative operation as an example, when $\mathfrak{f}(a) = 0.8$, this signifies element a demonstrates 80% incompatibility with the anti-fuzzy subgroupoid characteristics defined by $\mathfrak{f}$.

In this section, we define the subset relation and the intersection of two anti-fuzzy subgroupoids, based on the framework introduced by Atanassov (Atanassov, 1986)**.**

Definition 2.11. Let $\mathfrak{f}$ and $\mathfrak{d}$ be a fuzzy subset of a groupoid S. We say that $\mathfrak{f}$ is a non-membership subset of $\mathfrak{d}$, denoted by $\mathfrak{f} \tilde{\subseteq} \mathfrak{d}$, if their non-membership functions satisfy the following condition:

$$\mathfrak{f}(a) \geq \mathfrak{d}(a)$$

for all $a \in S$.

Definition 2.12. Let $\mathfrak{f}$ and $\mathfrak{d}$ be a fuzzy subset of a groupoid S. The intersection of $\mathfrak{f}$ and $\mathfrak{d}$, denoted by $\mathfrak{f} \tilde{\cap} \mathfrak{d}$, is defined by their non-membership functions as:

$$\mathfrak{f} \tilde{\cap} \mathfrak{d}(a) \stackrel{\text{def}}{=} \mathfrak{f}(a) \vee \mathfrak{d}(a)$$

for all $a \in S$.

RESEARCH METHODOLOGY

This study adopts a theoretical and deductive methodology. It begins by reviewing foundational concepts including groupoids, fuzzy sets, fuzzy subgroupoids, and anti-fuzzy subgroupoids. These concepts form the basis for constructing formal definitions and logical frameworks. The research proceeds by formulating propositions and proving them mathematically to explore the structural properties of lower-level subsets within anti-fuzzy subgroupoids. To support the validity of the theoretical results, illustrative examples and counterexamples are constructed and analyzed.

RESULTS AND DISCUSSION

The results of this study are presented systematically through a series of propositions. Each proposition is designed to examine specific mathematical properties of anti-fuzzy subgroupoid structures, particularly in relation to lower-level subsets. Before each proposition, all relevant symbols and notations are clearly introduced to ensure clarity and consistency. The deductive proofs provided aim to establish foundational relationships and behaviors within the framework of fuzzy algebraic systems.

Definition 3.1 Let $\mathfrak{f}$ be a fuzzy subset of a groupoid S. For any $\sigma \in [0,1]$, the lower-level subset of $\mathfrak{f}$, denoted by $\overline{\mathfrak{f}_\sigma}$, is defined by

$$\overline{\mathfrak{f}_\sigma} := \{a \in S | \mathfrak{f}(a) \leq \sigma\}.$$

Proposition 3.2 Let $\mathfrak{f}$ be a fuzzy subset of the groupoid S. For any threshold $\sigma \in [0,1]$, the lower-level subset $\overline{\mathfrak{f}_\sigma}$ satisfies the following properties:

1. Completeness at $\sigma = 1$:
$$\overline{\mathfrak{f}_1} = S.$$
2. Partition of the Universe:
For any $\sigma \in [0,1]$,
$$\mathfrak{f}_\sigma \cup \overline{\mathfrak{f}_\sigma} = S$$
3. Monotonicity concerning σ:
If $\sigma_1 < \sigma_2$, then:
$$\overline{\mathfrak{f}_{\sigma_1}} \subseteq \overline{\mathfrak{f}_{\sigma_2}}.$$

Proof

1. Let $a \in S$ be an arbitrary element. By the definition of the fuzzy subset $\mathfrak{f}$, the membership degree satisfies $\mathfrak{f}(a) \in [0,1]$. Since $\mathfrak{f}(a) \leq 1$ for all $a \in S$,

it follows that $a \in \overline{\mathfrak{f}_1}$. The preceding argument shows that every element of S is also an element of $\overline{\mathfrak{f}_1}$, which establishes the inclusion $S \subseteq \overline{\mathfrak{f}_1}$.

2. Take an arbitrary element $a \in S$. For any threshold $\sigma \in [0,1]$. There are two possible cases for the value of $\mathfrak{f}(a)$:

   Case 1: $\mathfrak{f}(a) \leq \sigma$. By the definition of the lower-level subset $\overline{\mathfrak{f}_\sigma}$, $a \in \overline{\mathfrak{f}_\sigma}$.

   Case 2: $\mathfrak{f}(a) > \sigma$. By the definition of the complementary level subset $\mathfrak{f}_\sigma$, $a \in \mathfrak{f}_\sigma$.

   Since a was arbitrary, we conclude that every element of S belongs to either $\overline{\mathfrak{f}_\sigma}$ or $\mathfrak{f}_\sigma$. Hence,

   $$S \subseteq \mathfrak{f}_\sigma \cup \overline{\mathfrak{f}_\sigma}.$$

   Conversely, take any element $a \in \mathfrak{f}_\sigma \cup \overline{\mathfrak{f}_\sigma}$. By the definition of union, a belongs to at least one of the two subsets $\overline{\mathfrak{f}_\sigma}$ or $\mathfrak{f}_\sigma$ both of which are subsets of S. Therefore,

   $$\mathfrak{f}_\sigma \cup \overline{\mathfrak{f}_\sigma} \subseteq S.$$

   By combining the two inclusions, we have

   $$\mathfrak{f}_\sigma \cup \overline{\mathfrak{f}_\sigma} = S.$$

   Therefore, the proof is complete.

   This result shows that the union of a lower-level subset and its complementary subset always cover the entire universe S. Intuitively, this reflects the partitioning nature of fuzzy sets, where each element either satisfies the threshold condition or its complement. Such a property is fundamental in fuzzy theory, as it ensures that no element is left unclassified, which is crucial for applications in decision-making and classification systems.

3. Recall the definition of the lower-level subset for any $\sigma \in [0,1]$:

   $$\overline{\mathfrak{f}_\sigma} = \{a \in S | \mathfrak{f}(a) \leq \sigma\}.$$

   Let $\sigma_1, \sigma_2 \in [0,1]$ such that $\sigma_1 < \sigma_2$.

   Take any arbitrary element $a \in \overline{\mathfrak{f}_{\sigma_1}}$. By definition of $\overline{\mathfrak{f}_{\sigma_1}}$, this means:

   $$\mathfrak{f}(a) \leq \sigma_1.$$

   Since $\sigma_1 < \sigma_2$, it follows that:

   $$\mathfrak{f}(a) \leq \sigma_1 < \sigma_2.$$

   By the transitivity of inequality:

   $$\mathfrak{f}(a) < \sigma_2.$$

   Therefore, $a \in \overline{\mathfrak{f}_{\sigma_2}}$ by the definition of $\overline{\mathfrak{f}_{\sigma_2}}$.

   Since the choice of $a \in \overline{\mathfrak{f}_{\sigma_1}}$ was arbitrary, this implies:

   $$\overline{\mathfrak{f}_{\sigma_1}} \subseteq \overline{\mathfrak{f}_{\sigma_2}}. \blacksquare$$

The family of lower-level subsets $\{\overline{\mathfrak{f}_\sigma} \mid \sigma \in [0,1]\}$ is monotonically increasing concerning the threshold σ. That is, for $\sigma_1 < \sigma_2$, the inclusion $\overline{\mathfrak{f}_{\sigma_1}} \subseteq \overline{\mathfrak{f}_{\sigma_2}}$ holds.

Proposition 3.3 A fuzzy subset $\mathfrak{f}$ of a groupoid S is an anti-fuzzy subgroupoid of S if and only if each nonempty lower-level subset $\overline{\mathfrak{f}_\sigma}$ of $\mathfrak{f}$ is a subgroupoid of S, for all $\sigma \in [0,1]$.

Proof. Suppose $\mathfrak{f}$ is an anti-fuzzy subgroupoid of S and $\overline{\mathfrak{f}_\sigma} \neq \emptyset$ for all $\sigma \in [0,1]$. Let $a, c \in \overline{\mathfrak{f}_\sigma}$. Then $\mathfrak{f}(a) \leq \sigma$ and $\mathfrak{f}(c) \leq \sigma$. Since $\mathfrak{f}(ac) \leq \mathfrak{f}(a) \vee \mathfrak{f}(c) \leq \sigma$, we have $ac \in \overline{\mathfrak{f}_\sigma}$. Hence $\overline{\mathfrak{f}_\sigma}$ is a subgroupoid of S.

Conversely, assume that each nonempty $\overline{\mathfrak{f}_\sigma}$ is a subgroupoid of S, for all $\sigma \in [0,1]$. Let if possible, $\mathfrak{f}(a_0c_0) > \mathfrak{f}(a_0) \vee \mathfrak{f}(c_0)$, for some $a_0, c_0 \in S$. Select $\sigma_0 \in [0,1]$ such that

$$\sigma_0 = \tfrac{1}{2}\big[\mathfrak{f}(a_0c_0) + \big(\mathfrak{f}(a_0) \vee \mathfrak{f}(c_0)\big)\big].$$

That is to say

$$\mathfrak{f}(a_0c_0) > \sigma > \big(\mathfrak{f}(a_0) \vee \mathfrak{f}(c_0)\big).$$

Consequently, we have $a_0c_0 \notin \overline{\mathfrak{f}_\sigma}$ for some $a_0, c_0 \in \overline{\mathfrak{f}_\sigma}$. The above argument is a contradiction, and so $\mathfrak{f}(ac) \leq \mathfrak{f}(a) \vee \mathfrak{f}(c)$ for all $a, c \in S$.

Therefore, $\mathfrak{f}$ is an anti-fuzzy subgroupoid of S. ■

Proposition 3.4 A fuzzy subset $\mathfrak{f}$ of a groupoid S is an anti-fuzzy subgroupoid of S if and only if each nonempty lower-level subset $\overline{\mathfrak{f}_\sigma}$ of $\mathfrak{f}$ is a subgroupoid of S, for all $\sigma \in [0,1]$.

Proof. Suppose $\mathfrak{f}$ is an anti-fuzzy subgroupoid of S and $\overline{\mathfrak{f}_\sigma} \neq \emptyset$ for all $\sigma \in [0,1]$. Let $a, c \in \overline{\mathfrak{f}_\sigma}$. Then $\mathfrak{f}(a) \leq \sigma$ and $\mathfrak{f}(c) \leq \sigma$. Since $\mathfrak{f}(ac) \leq \mathfrak{f}(a) \vee \mathfrak{f}(c) \leq \sigma$, we have $ac \in \overline{\mathfrak{f}_\sigma}$. Hence $\overline{\mathfrak{f}_\sigma}$ is a subgroupoid of S.

Proposition 3.5 Let $\mathfrak{f}$ and $\mathfrak{d}$ be fuzzy subsets of a groupoid S. If $\mathfrak{f}$ and $\mathfrak{d}$ are anti-fuzzy subgroupoids of S, then every nonempty lower-level subset $\overline{(\mathfrak{f} \widetilde{\cap} \mathfrak{d})_\sigma}$ is a subgroupoid of S for all $\sigma \in [0,1]$.

Proof. To prove the validity of Proposition 3.5, it suffices, based on Proposition 3.3, to show $\mathfrak{f} \cap \mathfrak{d}$ is an anti-fuzzy subgroupoid of S. Since $\mathfrak{f}$ and $\mathfrak{d}$ are anti-fuzzy subgroupoids of S, it follows from Definition 2.9 that for every $a, c \in S$, the following holds:

$$\begin{aligned}
\mathfrak{f} \widetilde{\cap} \mathfrak{d}(ac) &= \mathfrak{f}(ac) \vee \mathfrak{d}(ac) \\
&\leq \big(\mathfrak{f}(a) \vee \mathfrak{f}(c)\big) \vee \big(\mathfrak{d}(a) \vee \mathfrak{d}(c)\big) \\
&= \big(\mathfrak{f}(a) \vee \mathfrak{d}(a)\big) \vee \big(\mathfrak{f}(c) \vee \mathfrak{d}(c)\big) \\
&= \mathfrak{f} \widetilde{\cap} \mathfrak{d}(a) \vee \mathfrak{f} \widetilde{\cap} \mathfrak{d}(c)
\end{aligned}$$

Thus, $\mathfrak{f} \widetilde{\cap} \mathfrak{d}$ is an anti-fuzzy subgroupoid of S. ■

In general, the converse of Proposition 3.5 does not hold. To illustrate, consider the groupoid $S = \{a, c\}$ with the binary operation defined as follows:

$$aa = cc = a \text{ dan } ac = ca = c.$$

Let $\mathfrak{f}$ and $\mathfrak{d}$ be fuzzy subsets of S defined as:

$$\mathfrak{f}(a) = 0.2, \mathfrak{f}(c) = 0.4, \mathfrak{d}(a) = 0.3, \text{ and } \mathfrak{d}(c) = 0.1$$

such that:

$$\mathfrak{f}\breve{\cap}\mathfrak{d}(a) = 0.3 \text{ and } \mathfrak{f}\breve{\cap}\mathfrak{d}(c) = 0.4.$$

Consequently:

$$\mathfrak{f}\breve{\cap}\mathfrak{d}(aa) = \mathfrak{f}\breve{\cap}\mathfrak{d}(a) = 0.3 = \mathfrak{f}\breve{\cap}\mathfrak{d}(a) \vee \mathfrak{f}\breve{\cap}\mathfrak{d}(a),$$
$$\mathfrak{f}\breve{\cap}\mathfrak{d}(ac) = \mathfrak{f}\breve{\cap}\mathfrak{d}(c) = 0.4 = \mathfrak{f}\breve{\cap}\mathfrak{d}(a) \vee \mathfrak{f}\breve{\cap}\mathfrak{d}(c),$$
$$\mathfrak{f}\breve{\cap}\mathfrak{d}(ca) = \mathfrak{f}\breve{\cap}\mathfrak{d}(c) = 0.4 = \mathfrak{f}\breve{\cap}\mathfrak{d}(c) \vee \mathfrak{f}\breve{\cap}\mathfrak{d}(a),$$

and

$$\mathfrak{f}\breve{\cap}\mathfrak{d}(cc) = \mathfrak{f}\breve{\cap}\mathfrak{d}(a) = 0.3 < 0.4 = \mathfrak{f}\breve{\cap}\mathfrak{d}(c) \vee \mathfrak{f}\breve{\cap}\mathfrak{d}(c).$$

Therefore, $\mathfrak{f}\breve{\cap}\mathfrak{d}$ is an anti-fuzzy subgroupoid of S. However, $\mathfrak{d}$ is not an anti-fuzzy subgroupoid of S, since

$$\mathfrak{d}(cc) = \mathfrak{d}(a) = 0.3 \not\leq 0.1 = \mathfrak{d}(c) \vee \mathfrak{d}(c).$$

Based on these conditions, we derive the following result:

Proposition 3.6 Let $\mathfrak{f}$ and $\mathfrak{d}$ be fuzzy subsets of a groupoid S such that $\mathfrak{f} \breve{\subseteq} \mathfrak{d}$. If $\mathfrak{f}\breve{\cap}\mathfrak{d}$ is an anti-fuzzy subgroupoid of S, then $\mathfrak{f}$ is an anti-fuzzy subgroupoid of S.

Proof. Suppose $\mathfrak{f} \breve{\subseteq} \mathfrak{d}$. By Definition 2.11, we have $\mathfrak{f}(a) \geq \mathfrak{d}(a)$ for all $a \in S$. Hence, by Definition 2.12, for any $a \in S$,

$$\mathfrak{f}\breve{\cap}\mathfrak{d}(a) = \mathfrak{f}(a) \vee \mathfrak{d}(a) = \mathfrak{f}(a).$$

Thus,

$$\mathfrak{f}\breve{\cap}\mathfrak{d} = \mathfrak{f}.$$

As $\mathfrak{f}\breve{\cap}\mathfrak{d}$ is an anti-fuzzy subgroupoid of S, it follows that $\mathfrak{f}$ is an anti-fuzzy subgroupoid of S. ■

Proposition 3.7 Let K be a subgroupoid of a groupoid S. Then for any $\sigma \in (0,1)$ there exists an anti-fuzzy subgroupoid $\mathfrak{f}$ of S such that $\overline{\mathfrak{f}_\sigma} = K$.

Proof. Let $\mathfrak{f}$ be a fuzzy set of S defined by

$$\mathfrak{f}(a) = \begin{cases} 0, & a \in K \\ \sigma, & a \notin K \end{cases}$$

for any $a \in S$ where σ is a fixed number in $(0,1)$. By Definition 3.1, we have

$$\overline{\mathfrak{f}_\sigma} = \{a \in S | \mathfrak{f}(a) \leq \sigma\} = K.$$

Let $a, c \in S$. Then, by routine calculations, we have

$$\mathfrak{f}(ac) \leq \mathfrak{f}(a) \vee \mathfrak{f}(c).$$

Assume that $\mathfrak{f}(ac) > \mathfrak{f}(a) \vee \mathfrak{f}(c)$ for some $a, c \in S$. Since $\mathfrak{f}$ is two-valued, i.e., $|Im\, \mathfrak{f}| = 2$, $\mathfrak{f}(ac) = \sigma$ and $\mathfrak{f}(a) \vee \mathfrak{f}(c) = 0$ and hence $ac \notin K$ and $a, c \in K$. Since K is a subgroupoid of a groupoid S, $ac \in K$, a contradiction. Therefore, $\mathfrak{f}(ac) \leq \mathfrak{f}(a) \vee \mathfrak{f}(c)$ for any $a, c \in S$, i.e., $\mathfrak{f}$ be an anti-fuzzy subgroupoid of S. ■

Proposition 3.8 Let $\mathfrak{f}$ be an anti-fuzzy subgroupoid of a groupoid S. Then, the two lower subgroupoids $\overline{\mathfrak{f}_{\sigma_1}}$ dan $\overline{\mathfrak{f}_{\sigma_2}}$ (with $\sigma_1 < \sigma_2$) of $\mathfrak{f}$ are equal if and only if there is no element $a \in S$ such that $\sigma_1 < \mathfrak{f}(a) < \sigma_2$.

Proof. Let $\overline{\mathfrak{f}_{\sigma_1}} = \overline{\mathfrak{f}_{\sigma_2}}$. Then for any $a \in \overline{\mathfrak{f}_{\sigma_1}}$, we have that $a \in \overline{\mathfrak{f}_{\sigma_2}}$. That is, $\mathfrak{f}(a) \leq \sigma_1$ and $\mathfrak{f}(a) \leq \sigma_2$. Hence, there is no $a \in S$ such that $\sigma_1 < \mathfrak{f}(a) < \sigma_2$. Conversely, suppose there is no $a \in S$ such that $\sigma_1 < \mathfrak{f}(a) < \sigma_2$. Then we have $\mathfrak{f}(a) \leq \sigma_1$ and $\mathfrak{f}(a) < \sigma_2$. Therefore $a \in \overline{\mathfrak{f}_{\sigma_1}}$ such that $a \in \overline{\mathfrak{f}_{\sigma_2}}$. In other words, $\overline{\mathfrak{f}_{\sigma_1}} \subseteq \overline{\mathfrak{f}_{\sigma_2}}$. Since $\sigma_1 < \sigma_2$, by Proposition 3.2, we have $\overline{\mathfrak{f}_{\sigma_2}} \subseteq \overline{\mathfrak{f}_{\sigma_1}}$. Thus, $\overline{\mathfrak{f}_{\sigma_1}} = \overline{\mathfrak{f}_{\sigma_2}}$. ■

Proposition 3.9 Let S be a groupoid and let $\mathfrak{f}$ be an anti-fuzzy subgroupoid of a S. If $Im\,\mathfrak{f} = \{\sigma_1, \sigma_2, \cdots, \sigma_n\}$, where $\sigma_1 > \sigma_2 > \cdots > \sigma_n$, then the family of subgroupoids $\overline{\mathfrak{f}_{\sigma_i}}$ (for $i = 1, 2, \cdots, n$) constitutes all the lower-level subgroupoids of $\mathfrak{f}$.

Proof. Let $\sigma \in [0, 1]$ and $\sigma \notin Im\,\mathfrak{f}$.

1) If $\sigma_1 < \sigma$, then by Proposition 3.2, we have $\overline{\mathfrak{f}_{\sigma_1}} \subseteq \overline{\mathfrak{f}_{\sigma}}$. Since $\overline{\mathfrak{f}_{\sigma_1}} = S$, it follows that $\overline{\mathfrak{f}_{\sigma}} = S$, and thus $\overline{\mathfrak{f}_{\sigma_1}} = \overline{\mathfrak{f}_{\sigma}}$.
2) If $\sigma_i < \sigma < \sigma_{i+1}$ (for $i = 1, 2, \cdots, n-1$), then there is no $a \in S$ such that $\sigma \leq \mathfrak{f}(a) < \sigma_{i+1}$. By Proposition 3.7, we have $\overline{\mathfrak{f}_{\sigma_{i+1}}} = \overline{\mathfrak{f}_{\sigma}}$.

This shows that for any $\sigma \in [0, 1]$, the lower-level subgroupoid $\overline{\mathfrak{f}_{\sigma}}$ is in the set $\left\{\overline{\mathfrak{f}_{\sigma_i}} \,\middle|\, 1 \leq i \leq n\right\}$. ■

CONCLUSION

This study provides a formal characterization of anti-fuzzy subgroupoid structures within the framework of groupoids, extending fuzzy subgroup theory to algebraic systems that do not require identity or inverse elements. Through a deductive and axiomatic approach, the research establishes that every subgroupoid can be represented as a lower-level subset of a suitably constructed anti-fuzzy subgroupoid. Additionally, it demonstrates that two lower-level subsets are equal if and only if no element exists with a membership value strictly between the corresponding thresholds. These results offer a generalized framework for analyzing fuzzy structures in non-classical algebra and lay the groundwork for potential applications in uncertainty-based decision-making systems.

ACKNOWLEDGMENT

This research was financially supported by the Lambung Mangkurat University Research Grant, grant number 1725/UN8/LT/2025, for the year 2025.

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