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NON-STANDARD SCHEME DISCRETIZATION (NSFD) FOR COMMENSALISM SYMBIOSIS MODEL WITH HARVESTING IN COMMENSAL POPULATIONS

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ABSTRACT

Dynamic analysis on the model of commensalism symbiosis with the discretized Michaelis-Menten cropping by using different schemes to non-standard finite difference (NSFD) is the main focus in this article. The analysis is started by searching the equilibrium points with their existence terms and local stability with their stability terms. In this article, there are four equilibrium points. The points are the extinction point of both populations, the host extinction point, the commensal extinction point, and the point where both populations can coexist (the coexistence equilibrium point). The existence of a host extinction point and a point at which both populations can coexist depends on the conditions of existence that must be met. Among the four equilibrium points, the commensal extinction point and the coexistence equilibrium point are locally asymptotically stable provided that the specified stability conditions are met. In the final analysis, numerical simulations were performed using the 4^{th} order Runge–Kutta scheme for the continuous model and the NSFD scheme for the discrete model. The results show that the NSFD scheme offers greater flexibility in choosing the integration time step to ensure convergence to a feasible solution, outperforming the 4^{th} order Runge–Kutta scheme in this respect.

Keywords: Commensalism, discretization, local stability analysis, Michael-Menten.

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INTRODUCTION

These interactions include symbiosis, competition and predation. Symbiosis is a relationship between one living creature and another living creature. There are several types of symbiosis, namely commensalism, parasitism, mutualism, amensalism, and others. Commensalism symbiosis is a relationship, one living thing benefits (commensal) while the other party is neither benefited nor harmed (host). As an example in this context, mango tree with orchid as in figure 1. Orchid flowers benefit by residing on the mango tree for support, while the mango tree remains unaffected, as orchids are capable of producing their own food through photosynthesis. Neither the mango tree gain benefit because in the making of its own food, the mango tree is not assisted by the orchid (Edessa, 2018).

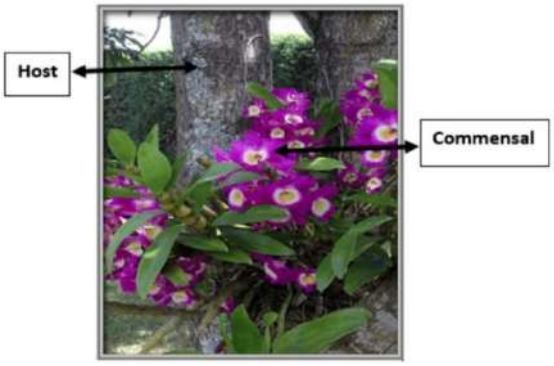


Figure 1. Example of symbiosis of commensalism
https://kependidikan.com/simbiosis-komensalisme/simbiosis-komensalisme-anggrek-500px/ (*Simbiosis Komensalisme Anggrek*, n.d.)

Dynamics analysis of the symbiotic model is an important concept in mathematics. In 1838, Pierre Verhulst was the first to introduce the logistic growth model. The logistic model was formed to carry out the process of limiting the population grew too large. This limitation is achieved by considering the environmental carrying capacity. This carrying capacity is the maximum capacity of the environment that can support a given number of individuals in the long term (Murray, 2002).

Various studies have examined mathematical models of symbiosis. One example is Chen's research, which explores mathematical models of commensalism. In Chen's research, the description of commensal and host growth used a logistic model. Chen also introduced a model for harvesting is the Michaelis-Menten type (Chen, 2019). Furthermore, in 2020 Wei's conducted research on the mathematical model of amensalism added with the Alle effect (Wei et al., 2020).

Research on symbiotic models continues to progress with the addition of several assumptions to make the model more complex and realistic. This development includes the use of various response functions. One example of the response function used is the research by Bairagi & Biswas which uses the Beddington de Angeli response function (Bairagi & Biswas, 2016). In addition, there is also, (Chen & Wu, 2017) research which uses the the non-monotonic response function, (Ongun & Ozdogan, 2017) research which uses the Holling type II response function with the Allee effect, (Chen, 2018) research which uses the ratio-dependent response function with the Allee effect, and (Ye et al., 2019) research which uses the Holling type I response function with the Allee effect.

The Michaelis-Menten harvesting model describes a scenario in which the harvesting rate rises with population density but approaches a maximum limit due to saturation effects. One example is Huo's research, which discusses predator plowing, where predator populations are harvested using the Michaelis Menten type (Hu & Cao, 2017). In addition, there is also research on, Gause's predator-prey model with gestation delay and Michaelis-Menten type prey harvesting and is

analyzed considering Holling's type III functional response (Liu & Jiang, 2018), two-species amensalism model with Michaelis–Menten type harvesting for the amensal species (Liu et al., 2018), predator–prey model with Michaelis–Menten functional response and divided the prey population in two subpopulations: susceptible and infected prey (Saha et al., 2018), prey–predator model with Michaelis–Menten type predator harvesting (Fattahpour et al., 2019), almost periodic commensal symbiotic model with nonlinear harvesting on a time scale (Xue et al., 2019), a Lotka–Volterra predator–prey model combining Michaelis–Menten-type harvesting and prey fear effects (Lai et al., 2020), Leslie–Gower diffusive predator-prey model with Michaelis-Menten type harvesting in prey populations (Zuo et al., 2020), a prey-predator-scavenger model involving Michaelis-Menten type harvesting in both predator and scavenger species by Satar & Naji (Satar & Naji, 2022).

An analytical solution is difficult to obtain, requiring a numerical approach. One such method for solving continuous models is through discretization. For example, Mickens discretization as the one that has been done (Tassaddiq et al., 2020), (Ongun & Ozdogan, 2017), (Shabbir et al., 2019). The discrete model resulting from the discretization process approaches the solution of the continuous model. Their results indicate that, on the discrete model show that the dynamics of the discrete model are richer when compared to the continuous model.

Based on the description of previous research, this study will discretize the commensalism symbiotic model with harvesting in commensal species developed by (B. Chen, 2019). The results of this study are in the form of equilibrium points, existence conditions, and local stability of the equilibrium point. In this local stability, equilibrium stability may be either asymptotically stable or unstable depends on the conditions. In the final analysis, numerical simulations are performed using the 4^{th} order Runge–Kutta for the continuous model and the NSFD scheme for the discrete model. This numerical analysis will demonstrate that the NSFD scheme provides greater flexibility in selecting the integration time step to ensure convergence to a feasible solution, thus outperforming the 4^{th} order Runge–Kutta.

METHOD

The several stages and analysis procedures carried out in this study are as shown below.

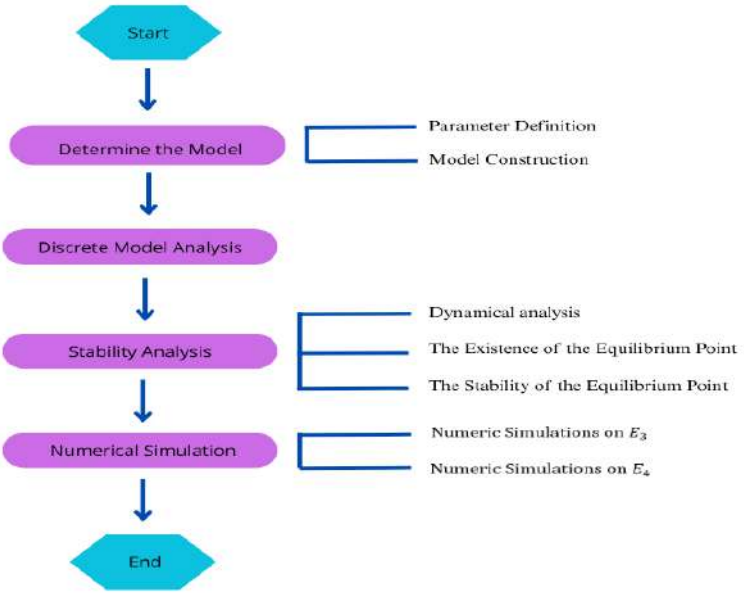


Figure 2. *Stability Analysis Procedure*

1. Determine the Model

Chen (2019) studied a two-dimensional commensalism model with harvesting using Michaelis-Menten. The following is a model of commensalism with the Michaelis-Menten type of harvesting:

$$\begin{aligned}\frac{dx}{dt} &= r_1 x\left(1-\frac{x}{k_1}+a\frac{y}{k_1}\right)-\frac{qEx}{m_1E+m_2x},\\ \frac{dy}{dt} &= r_2 y\left(1-\frac{y}{k_2}\right),\end{aligned} \tag{1}$$

It can be seen that

$x(t)$: the density of the commensal species
$y(t)$: the density of the host species.
r_1 : commensal intrinsic growth
r_2 : host intrinsic growth
k_1 : carrying capacities in the commensal
k_2 : carrying capacities in the host
a : parameter of interaction between commensal and host
q : the catchability coefficient harvest
E : the parameter of catching effort/energy used to harvest
m_1 : Response parameters of harvesting absorption to effort
m_2 : Harvest absorption response parameters to population

It is assumed that all parameters are positive

2. Discrete Model and Stability Analysis

An analysis of the discrete model within the system of equations (1) uses the following definitions and lemmas. Where lemma 1 is a way to discretize the NSFD, definition 1 is to describe the equilibrium point, and lemma 2 is to determine the

stability of the equilibrium point.

Lemma 1.

(Mickens, 2005) For example the discretization using a non-standard discretization scheme for the rate of change in the number of commensal populations as follows.

a. The first derivative discretization with a different forward approach

$$\frac{dx}{dt} = \frac{x_{n+1} - x_n}{\Delta t},$$

that $\Delta t > 0$ is the time-step.

b. The non-linear form of the differential equation is approximated by the non-local discrete form. In this case, if there is a non-linear root form x^2 it will be converted into the form $x_n x_{n+1}$, while for xy it will be changed to $x_n y_n$ or $x_n y_{n+1}$ or $x_{n+1} y_n$.

Definition 1.

(Mickens & Elyadi, 1997) A point x^* which is in the domain is said to be the equilibrium point of a system of different equations $fx_{n+1} = f(x_n)$ if $f(x^*) = x^*$. It is assumed that the eigenvalues of the Jacobi matrix $Df(x^*)$ are λ_1 and λ_2 so:

Lemma 2.

(Abdelaziz et al., 2018) It is assumed that the eigenvalues of the Jacobi matrix $Df(x^*)$ are λ_1 and λ_2 such that:

a. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$ then the equilibrium point of $Df(x^*)$ is locally asymptotically stable.
b. If $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$ then the equilibrium point of $Df(x^*)$ is saddle unstable.
c. If $|\lambda_1| = 1$ or $|\lambda_2| = 1$ then the equilibrium point of $Df(x^*)$ is non-hyperbolic.
d. If $|\lambda_1| > 1$ and $|\lambda_2| > 1$ or $|\lambda_{1,2} = 1|$ then the equilibrium point of $Df(x^*)$ is unstable.

3. Numerical Simulation

To validate the analytical results, numerical simulations were conducted. In this study, the simulations were performed using MATLAB version R2015b.

RESULTS AND DISCUSSION

1. Dynamic analysis of the model

Analysis of the system's dynamics, initially formulated in continuous form as shown in Equation (1), requires transformation into a format suitable for numerical investigation. The continuous model is therefore reformulated into a discrete system through a discretization process. Standard discretization methods, such as the classical Euler scheme, often fail to preserve essential qualitative features of the original system. These include the stability of equilibrium points and the long-term behavior of solutions, potentially leading to discrepancies between analytical and

numerical results.

A nonstandard finite difference scheme is adopted in this study to address these limitations. This approach preserves the dynamic consistency and structural integrity of the original system by modifying conventional derivative approximations and incorporating nonlinear terms in a specially constructed rational form. The application of this nonstandard scheme results in a discrete-time iterative system that accurately reflects the key dynamics of the original model. The resulting system is expressed in Equation (2) as follows:

$$\begin{aligned} \frac{x_{n+1}-x_n}{\Delta t} &= r_1 x_n \ - \frac{r_1 x_n \ x_{n+1}}{k_1} + r_1 a \frac{x_n \, y_n}{k_1} - \frac{qEx_n}{m_1 E + m_2 x_n}, \\ \frac{y_{n+1}-y_n}{\Delta t} &= r_2 y_n \ - \frac{r_2}{k_2} y_n \ y_{n+1}, \end{aligned} \tag{2}$$

Equation (2) presents the discretized form of the original continuous system using a nonstandard finite difference scheme. This formulation includes implicit terms, such as x_{n+1} and y_{n+1}, appearing on both sides of the equations, which complicates direct numerical evaluation and analysis. To facilitate further analysis, particularly the investigation of equilibrium points and system stability, Equation (2) is algebraically manipulated into an explicit recursive form. This transformation isolates x_{n+1} and y_{n+1} on the left-hand side as functions of the known quantities x_n and y_n, resulting in a fully discrete-time dynamical system.

The explicit system is given in Equation (3), where the updated states x_{n+1} and y_{n+1} are expressed as nonlinear rational functions of the current states. This new representation, denoted as $f(x_n, y_n)$ and $g(x_n, y_n)$, provides a more suitable form for both analytical treatment and numerical simulations.

$$\begin{cases} x_{n+1} = \frac{x_n \ (k_1 m_1 E + k_1 m_2 x_n \ + w(k_1 + a y_n \ )(m_1 E + m_2 x_n \ ) - z E k_1)}{(m_1 E + m_2 x_n \ )(k_1 + w x_n \ )} = f(x_n, y_n), \\ y_{n+1} = \frac{y_n \ k_2 (d+1)}{(k_2 + d y_n \ )} = g(x_n, y_n), \end{cases} \tag{3}$$

The application of the nonstandard discretization scheme leads to a new discrete system, represented in Equation (3). This system consists of two recursive equations that define the evolution of state variables $x_n$and y_n over discrete time steps, with each update depending nonlinearly on the previous state. The parameters $d = \Delta t r_2, w = \Delta t r_1$ and $z = \Delta t q$ are introduced to simplify notation and capture the effects of the discretization step size Δt on the system dynamics. Identification of equilibrium points requires determining constant solutions (x^*, y^*) such that the system remains unchanged under iteration. This condition implies that the system satisfies: $x_{n+1} = x_n = x^*$ and $y_{n+1} = y_n = y^*$

Substituting these fixed-point conditions into the discrete system in Equation (3) yields a coupled nonlinear system, shown in Equation (4). The resulting algebraic equations define the equilibrium conditions for the discrete system and serve as the basis for further analytical investigations, including stability analysis and bifurcation behavior.

$$\begin{cases} x^* = \frac{x^*(k_1m_1E+k_1m_2x^*+w(k_1+ay^*)(m_1E+m_2x^*)-zEk_1)}{(m_1E+m_2x^*)(k_1+wx^*)} = f(x^*, y^*), \\ y^* = \frac{y^*k_2(d+1)}{(k_2+dy^*)} = g(x^*, y^*). \end{cases} \quad (4)$$

By solving the system of equation (3), four equilibrium points are obtained as follows.

$$E_1 = (0,0), E_2 = ({x_2}^*, 0), E_3 = (0, k_2), E_4 = ({x_4}^*, k_2),$$

that,

${x_2}^* = \frac{-w(m_1E-m_2k_1)\pm\sqrt{{D_2}^*}}{2m_2w}$,

${D_2}^* = \big(w(m_1E - m_2k_1)\big)^2 - 4\big(m_2wEk_1(z - m_1w)\big)$,

${x_4}^* = \frac{-w(m_1E-m_2k_1-ak_2m_2)\pm\sqrt{{D_4}^*}}{2m_2w}$,

${D_4}^* = \big(w(m_1E - m_2k_1 - ak_2m_2)\big)^2 - 4\big(m_2wE(zk_1 - m_1k_1w - ak_2m_1w)\big)$.

2. The Existence of the Equilibrium Point

The four equilibrium points above exist if they can satisfy following Theorem 1.

Theorem 1. If system solution of equations (4) is E_i, $i = 1,2,3,4$, then $E_i \geq 0$ is the equilibrium point of the system of equation (3) which satisfies the following existence conditions.

a. E_1 and E_3 is the equilibrium point of the system of equations (3).
b. E_2 is the equilibrium point of the system of equations (3) if meets these requirements
   $\sqrt{{D_2}^*} \geq 0$ and
   i. $Ek_1(z - m_1w) < 0$, or
   ii. $Ek_1(z - m_1w) = 0$ and $w(m_1E - m_2k_1) < 0$, or
   iii. $Ek_1(z - m_1w) > 0$ and $w(m_1E - m_2k_1) < 0$.
c. E_4 is the equilibrium point of the system of equations (3) if fulfill $\sqrt{{D_4}^*} \geq 0$ and
   i. $E(zk_1 - m_1k_1w - ak_2m_1w) < 0$, or
   ii. $E(zk_1 - m_1k_1w - ak_2m_1w) = 0$ and $w(m_1E - m_2k_1 - ak_2m_2) < 0$, or
   iii. $E(zk_1 - m_1k_1w - ak_2m_1w) > 0$ and $w(m_1E - m_2k_1 - ak_2m_2) < 0$.

Proof. In accordance with real-world conditions, population values are always non-negative; therefore, the solution of system (4) must also remain non-negative.

a. E_1 and E_3 are always non-negative, so that E_1 and E_3 is the equilibrium point of the system of equations (3).

b. If $\sqrt{D_2}^* \geq 0$, then has

i. one solution of non-negative which is $E_2 = \left(x_2{}^* = \frac{-w(m_1E-m_2k_1)+\sqrt{D_2}^*}{2m_2w} > 0, 0\right)$, or

ii. two non-negative solutions which is $E_1 = (0, 0)$ and $E_2 = \left(x_2{}^* = \frac{w(m_1E-m_2k_1)}{m_2w} > 0, 0\right)$, or

iii. two non-negative solutions which are $E_{2+} = \left(x_{2+}{}^* = \frac{-w(m_1E-m_2k_1)+\sqrt{D_2}^*}{2m_2w} > 0, 0\right)$ and $E_{2-} = \left(x_{2-}{}^* = \frac{-w(m_1E-m_2k_1)-\sqrt{D_2}^*}{2m_2w} > 0, 0\right)$.

Because the value of $x_2{}^* > 0$ and $y_2{}^* = 0$, then E_2 is the equilibrium point of the system of equations (3).

c. If $\sqrt{D_4}^* \geq 0$, then has

i. one non-negative solution which is $E_4 = \left(x_2{}^* = \frac{-w(m_1E-m_2k_1-ak_2m_2)+\sqrt{D_4}^*}{2m_2w} > 0, k_2\right)$, or

ii. two non-negative solutions which are $E_1 = (0, 0)$ and $E_4 = \left(x_2{}^* = \frac{w(m_1E-m_2k_1-ak_2m_2)}{m_2w} > 0, k_2\right)$, or

iii. two non-negative solutions which are $E_{4+} = \left(x_{2+}{}^* = \frac{-w(m_1E-m_2k_1-ak_2m_2)+\sqrt{D_4}^*}{2m_2w} > 0, k_2\right)$ and $E_{4-} = \left(x_{2-}{}^* = \frac{-w(m_1E-m_2k_1-ak_2m_2)-\sqrt{D_4}^*}{2m_2w} > 0, k_2\right)$.

Because the value of $x_4{}^* > 0$ and $y_4{}^* = k_2 > 0$, then E_4 is the equilibrium point of the system of equations (3).■

3. The Stability of the Equilibrium Point

In order to understand the behavior of the model around the equilibrium points E_i, where $i = 1,2,3,4$ it is used linearization to the model in the system of equation (3). Furthermore, by linearizing system (3) around the equilibrium point, the Jacobian matrix is obtained as follows.

$$J(x^*, y^*) = \begin{bmatrix} \frac{k_1{}^2+wk_1(k_1+ay^*)}{(k_1+wx^*)^2} + \frac{zEk_1\left(m_2wx^{*2}-m_1Ek_1\right)}{(m_1E+m_2x^*)^2(k_1+wx^*)^2} & \frac{wax^*}{k_1+wx^*} \\ 0 & \frac{k_2{}^2(1+d)}{(k_2+dy^*)^2} \end{bmatrix}.$$

The eigenvalues of matrix of $J(x^*, y^*)$ is obtained by $|J(x^*, y^*) - \lambda I| = 0$ which is

$$\lambda_1 = \frac{{k_1}^2 + wk_1(k_1 + ay^*)}{(k_1 + wx^*)^2} + \frac{zEk_1\left(m_2 wx^{*2} - m_1 Ek_1\right)}{\left((m_1 E + m_2 x^*)(k_1 + wx^*)\right)^2},$$

$$\lambda_2 = \frac{{k_2}^2(1+d)}{(k_2 + dy^*)^2}.$$

Based on Lemma 2, in determining stability E_i, $i = 1,2,3,4$ from the system of equations (3) expressed in the following theorem form.

Theorem 2. Equilibrium point E_1 has the following stability conditions

i. If $\Delta t > 0$, then equilibrium point E_1 is unstable.

Proof. The Eigenvalues of $J(E_1)$ matrix is obtained from $\lambda_1 = \frac{m_1 + m_1 w - z}{m_1}$ and $\lambda_2 = 1 + d$. It is known that the parameter value is positive, so that $|\lambda_2| > 1$. If $\Delta t > 0$, then using algebraic operations will be obtained $m_1 + \Delta t(m_1 r_1 - q) < m_1$ or $m_1 + \Delta t(m_1 r_1 - q) > m_1$ so that, $|\lambda_1| < 1$ or $|\lambda_1| > 1$. It is evident, however, that theorem 2 fulfills Lemma 2 (b). ■

Theorem 3. Equilibrium point E_2 has two stability conditions as follows

i. If $\Delta t_1 > 0$ or $\Delta t_2 > \frac{(m_1 E + m_2 x_n)^2(2k_1 x_n r_1 - r_1 {k_1}^2) + q{k_1}^2 m_1 E^2}{qEk_1 m_2 r_1 {x_n}^2 - {x_n}^2 {r_1}^2 (m_1 E + m_2 x_n)^2}$, then equilibrium point E_2 is unstable.

ii. If $\Delta t_1 > 0$ or $\Delta t_2 > \frac{(m_1 E + m_2 x_n)^2(r_1 {k_1}^2 - 2k_1 x_n r_1) - q{k_1}^2 m_1 E^2}{{x_n}^2 {r_1}^2 (m_1 E + m_2 x_n)^2 - qEk_1 m_2 r_1 {x_n}^2}$, then equilibrium point E_2 is saddle unstable.

Proof. The Eigenvalues of $J(E_2)$ matrix is obtained from $\lambda_1 = \frac{{k_1}^2 + w{k_1}^2}{(k_1 + w{x_2}^*)^2} + \frac{zEk_1\left(m_2 w{x_2}^{*2} - m_1 Ek_1\right)}{\left((m_1 E + m_2 {x_2}^*)(k_1 + w{x_2}^*)\right)^2}$ and $\lambda_2 = 1 + d$. It is known that the parameter value is positive, so that $|\lambda_2| > 1$. If $\Delta t_1 > 0$ or $\Delta t_2 > \frac{(m_1 E + m_2 x_n)^2(2k_1 x_n r_1 - r_1 {k_1}^2) + q{k_1}^2 m_1 E^2}{qEk_1 m_2 r_1 {x_n}^2 - {x_n}^2 {r_1}^2 (m_1 E + m_2 x_n)^2}$, then using algebraic operations will be obtained $(qEk_1 m_2 r_1 {x_n}^2)\Delta t^2 + \left(r_1 {k_1}^2 (m_1 E + m_2 x_n)^2 - q{k_1}^2 m_1 E^2\right)\Delta t + {k_1}^2 (m_1 E + m_2 x_n)^2 > (m_1 E + m_2 x_n)^2 (k_1 + \Delta t x_n r_1)^2$ so that, $|\lambda_1| > 1$. If $\Delta t_1 > 0$ or $\Delta t_2 > \frac{(m_1 E + m_2 x_n)^2(r_1 {k_1}^2 - 2k_1 x_n r_1) - q{k_1}^2 m_1 E^2}{{x_n}^2 {r_1}^2 (m_1 E + m_2 x_n)^2 - qEk_1 m_2 r_1 {x_n}^2}$, then using algebraic operations will be obtained $(qEk_1 m_2 r_1 {x_n}^2)\Delta t^2 + \left(r_1 {k_1}^2 (m_1 E + m_2 x_n)^2 - q{k_1}^2 m_1 E^2\right)\Delta t + {k_1}^2 (m_1 E + m_2 x_n)^2 < (k_1 + \Delta t x_n r_1)^2$ so that, $|\lambda_1| < 1$.
It is evident, however, that theorem 3 fulfills Lemma 2 (b) and (d). ■

Theorem 4. Then equilibrium point E_3 has stability conditions as follows

i. If $\Delta t_1 > 0$ and $\Delta t_2 > -\frac{1}{r_2}$, then equilibrium point E_3 is local asymptotically stable.

Proof. The Eigenvalue of $J(E_3)$ matrix is obtained from $\lambda_1 = \frac{m_1 k_1 + m_1 \Delta t r_1 (k_1 + ak_2) - qk_1 \Delta t}{m_1 k_1}$ and $\lambda_2 = \frac{{k_2}^2 (1 + \Delta t r_2)}{(k_2 + \Delta t r_2 k_2)^2}$. It is known that the

parameter value is positive, so that $|\lambda_2| < 1$, if ${k_2}^2(1 + \Delta t r_2) < (k_2 + \Delta t r_2 k_2)^2$ by using algebraic operations will be obtained ${k_2}^2(1 + \Delta t r_2) < (k_2 + \Delta t r_2 k_2)^2$ then you get $\Delta t_2 > -\frac{1}{r_2}$. If $\Delta t_1 > 0$, then using algebraic operations will be obtained $m_1 k_1 + m_1 \Delta t r_1(k_1 + a k_2) - q k_1 \Delta t < m_1 k_1$so that, $|\lambda_1| < 1$. It is evident, however, that theorem 4 fulfills Lemma 2(a). ■

Theorem 5. Then equilibrium point E_4 has two stability conditions as follows

i. If $\Delta t_1 > 0$ or $\Delta t_2 > \frac{(m_1 E + m_2 x_n)^2(2k_1 x_n\, r_1 - r_1 {k_1}^2 - r_1 a y_n\, k_1) + q {k_1}^2 m_1 E^2}{qE k_1 m_2 r_1 {x_n}^2 - {x_n}^2 {r_1}^2 (m_1 E + m_2 x_n)^2}$ and $\Delta t_3 > -\frac{1}{r_2}$, then equilibrium point E_4 is saddle unstable.

ii. If $\Delta t_1 > 0$ or $\Delta t_2 > \frac{(m_1 E + m_2 x_n)^2(r_1 {k_1}^2 + r_1 a y_n\, k_1 - 2k_1 x_n\, r_1) - q {k_1}^2 m_1 E^2}{{x_n}^2 {r_1}^2 (m_1 E + m_2 x_n)^2 - qE k_1 m_2 r_1 {x_n}^2}$and $\Delta t_3 > -\frac{1}{r_2}$, then equilibrium point E_4 is local asymptotically stable.

Proof. Eigenvalue of $J(E_4)$ matrix is obtained from $\lambda_1 = \frac{{k_1}^2 + w k_1(k_1 + a k_2)}{(k_1 + w {x_4}^*)^2} + \frac{zE k_1\left(m_2 w {x_4}^{*2} - m_1 E k_1\right)}{\left((m_1 E + m_2 {x_4}^*)(k_1 + w {x_4}^*)\right)^2}$ and $\lambda_2 = \frac{{k_2}^2(1 + \Delta t r_2)}{(k_2 + \Delta t r_2 k_2)^2}$. It is known that the parameter value is positive, so that $|\lambda_2| < 1$, if ${k_2}^2(1 + \Delta t r_2) < (k_2 + \Delta t r_2 k_2)^2$ by using algebraic operations will be obtained ${k_2}^2(1 + \Delta t r_2) < (k_2 + \Delta t r_2 k_2)^2$ then get $\Delta t_2 > -\frac{1}{r_2}$. If $\Delta t_1 > 0$ or $\Delta t_2 > \frac{(m_1 E + m_2 x_n)^2(2k_1 x_n\, r_1 - r_1 {k_1}^2 - r_1 a y_n\, k_1) + q {k_1}^2 m_1 E^2}{qE k_1 m_2 r_1 {x_n}^2 - {x_n}^2 {r_1}^2 (m_1 E + m_2 x_n)^2}$, then using algebraic operations will be obtained $(qE k_1 m_2 r_1 {x_n}^2)\Delta t^2 + \left(r_1 {k_1}^2 (m_1 E + m_2 x_n)^2 + r_1 a y_n\, k_1 (m_1 E + m_2 x_n)^2 - q {k_1}^2 m_1 E^2\right)\Delta t + {k_1}^2 (m_1 E + m_2 x_n)^2 > (m_1 E + m_2 x_n)^2 (k_1 + \Delta t x_n\, r_1)^2$so that, $|\lambda_1| > 1$. If $\Delta t_1 > 0$ or $\Delta t_2 > \frac{(m_1 E + m_2 x_n)^2(r_1 {k_1}^2 + r_1 a y_n\, k_1 - 2k_1 x_n\, r_1) - q {k_1}^2 m_1 E^2}{{x_n}^2 {r_1}^2 (m_1 E + m_2 x_n)^2 - qE k_1 m_2 r_1 {x_n}^2}$, then using algebraic operations will be obtained $(qE k_1 m_2 r_1 {x_n}^2)\Delta t^2 + \left(r_1 {k_1}^2 (m_1 E + m_2 x_n)^2 + r_1 a y_n\, k_1 (m_1 E + m_2 x_n)^2 - q {k_1}^2 m_1 E^2\right)\Delta t + {k_1}^2 (m_1 E + m_2 x_n)^2 < (k_1 + \Delta t x_n\, r_1)^2$so that, $|\lambda_1| < 1$. It is evident, however, that theorem 3 fulfills Lemma 2 (a)-(b). ■

E_1 and E_2 equilibrium points are the points that do not have stable possibility because of the existing structural, while E_3 and E_4 equilibrium points are the points that allow the local asymptotically stable if it fulfills the specified terms. It means that the *host* population has no possibility to be extinct or both populations will have a contiguous life, in other words, they will not be extinct simultaneously.

4. Numeric Simulation

Numerical simulations of system (3) are performed in this section using Matlab software using two schemes: NSFD and 4th-order Runge-Kutta. These numerical simulations aim to compare the consistency of system stability using the two methods. The parameter values and the initial values used in the numerical simulations in this paper are assumed hypothetically due to the lack of real-world

data. Therefore, the parameter values are.

Table 1. Parameter Values.

Parameter	Simulation 1	Simulation 2
r_1	0.9	1
r_2	5	5
k_1	1	1
k_2	1	1
a	0.9	1
q	5	2
E	1	1
m_1	2	1
m_2	1	1

Table 2. Initial Values.

Initial Values	Figure 3 and 4			Figure 5			Figure 6
$x(0)$	3	0.9	1.5	5	0.3	0.7	1
$y(0)$	0.5	0.9	0.1	0.1	3	2	0.1

Some numerical simulation that appropriate with the analysis results explained before will be given in this stage. This numerical simulation is used to observe the dynamic interactions between the commensal and host species over time. The numeric simulation that shows local stability of E_3 and E_4.

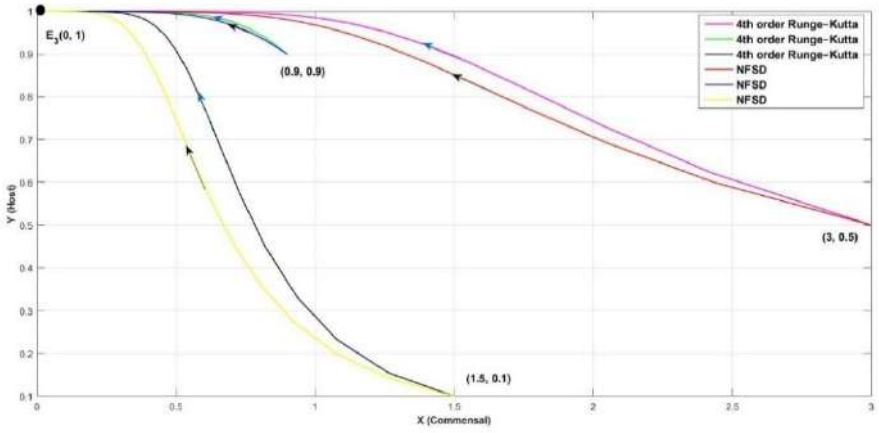


Figure 3. Numerical Simulation on E_3 with $\Delta t = 0.1$ using: 4th order Runge–Kutta scheme and NFSD scheme

To illustrate the behavior of the solution related to the stability of the coexistence equilibrium point E_4, the parameter values listed in Table 1 (Simulation 1) and the initial values in Table 2 are used. With these parameters, the stability condition of the equilibrium point $E_3(0,1)$ is maintained. This result is consistent with the numerical simulation shown in Figure 3, using a step size of $\Delta t = 0.1$.

When, $\Delta t = 0.1 > \Delta t_1 = 0$ and $\Delta t = 0.1 > \Delta t_2 = -0.2$, so $\Delta t = 0.1$ satisfies the stability condition of the equilibrium point E_3.

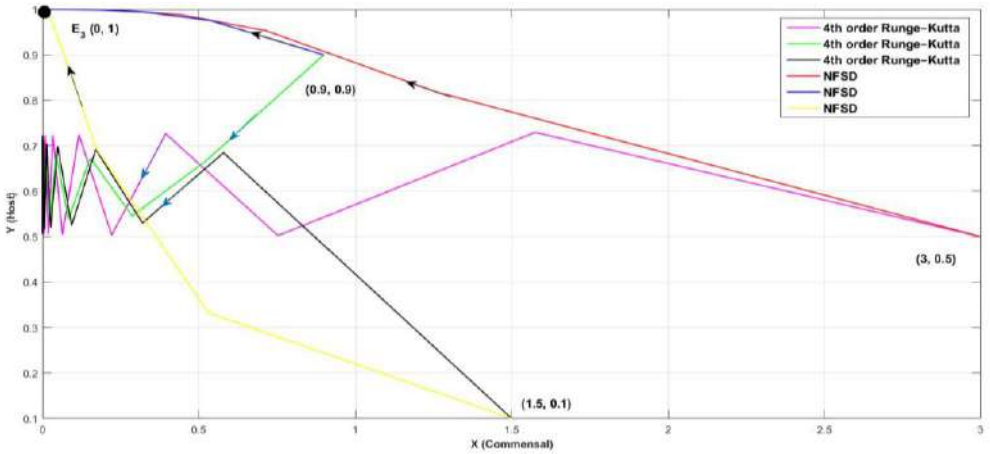
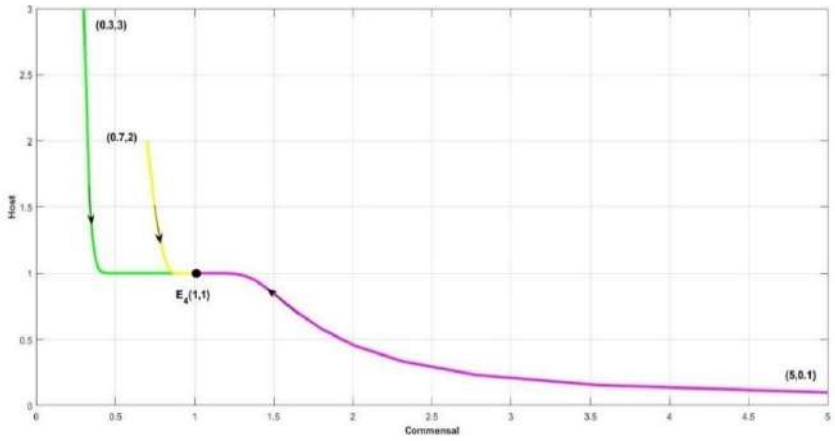
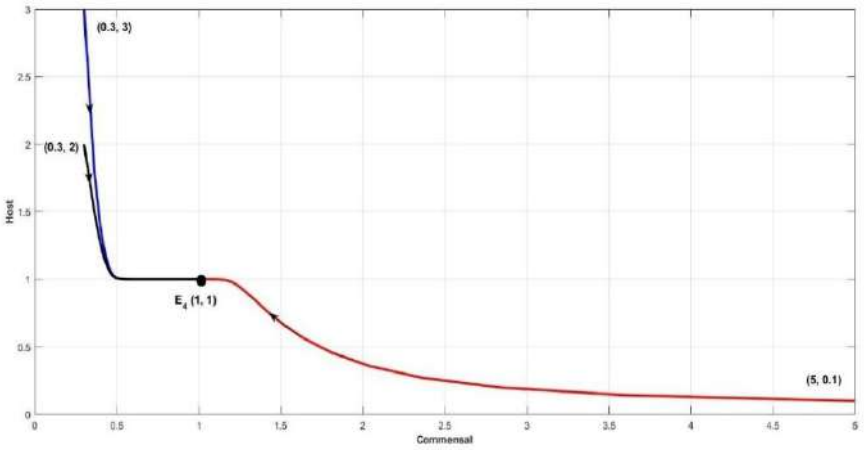


Figure 4. Numerical Simulation on E_3 with $\Delta t = 1$ using: 4th order Runge–Kutta scheme and NFSD scheme

When the step size is increased to $\Delta t = 1$, it satisfies the stability condition of the equilibrium point E_3, since $\Delta t = 1 > \Delta t_1 = 0$ and $\Delta t = 1 > \Delta t_1 = -0.2$. Under this condition, the behavior of the system's solution begins to change, as shown in Figure 6. With the initial values chosen near the equilibrium point E_3, the 4th order Runge–Kutta scheme produces an infeasible solution. The solution tends to converge to negative values. In contrast, the solution obtained using the NFSD scheme remains consistent and successfully preserves the stability of E_3. Subsequently, several parameter values from Simulation 1 (Table 1) were modified. The parameters r_1 and a were increased by 10%, while q was decreased by 60% and $m_1$was reduced by 50%, as presented in Simulation 2 (Table 1).



(a)



(b)

Figure 5. Numerical Simulation on E_4 with $\Delta t = 0.1$ using: (a). 4th order Runge–Kutta scheme and (b). NFSD scheme

To illustrate the behavior of the solution related to the stability of the coexistence equilibrium point E_4, the parameter values listed in Table 1 (Simulation 2) and the initial values in Table 2 are used. With these parameters, the stability condition of the equilibrium point $E_4(1,1)$ is maintained. This result is consistent with the numerical simulation shown in Figure 5, using a step size of $\Delta t = 0.1$. When, $\Delta t = 0.1 > \Delta t_3 = -0.2$ and $\Delta t = 0.1 > \Delta t_1 = 0$ or $\Delta t = 0.1 > \Delta t_2 = -1$, so $\Delta t = 0.1$ satisfies the stability condition of the equilibrium point E_4.

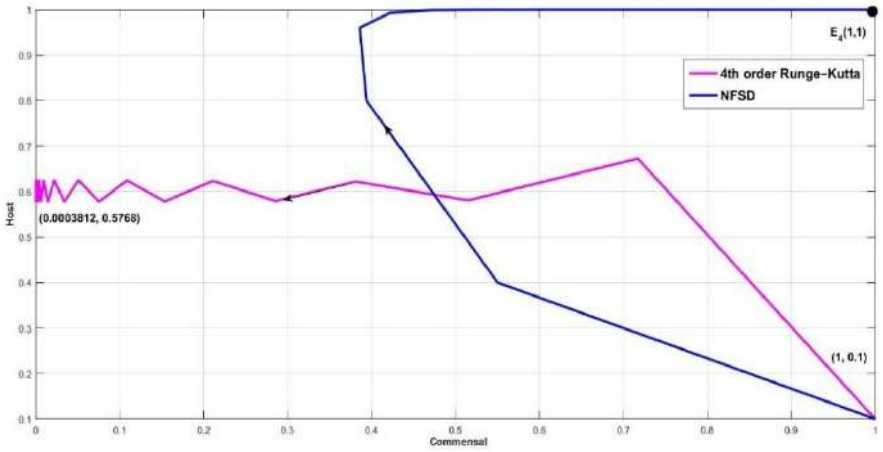


Figure 6. Numerical Simulation on E_4 with $\Delta t = 1$ using: 4th order Runge–Kutta scheme and NFSD scheme

When the step size is increased to $\Delta t = 1$, it satisfies the stability condition of the equilibrium point E_4, since $\Delta t = 1 > \Delta t_3 = -0.2$ and $\Delta t = 1 > \Delta t_1 = 0$ or $\Delta t = 1 > \Delta t_2 = -1$. Under this condition, the behavior of the system's solution begins to change, as shown in Figure 6. With the initial values chosen near the equilibrium point E_4, the 4th order Runge–Kutta scheme produces an infeasible

solution.

The solution tends to converge to negative values. In contrast, the solution obtained using the NFSD scheme remains consistent and successfully preserves the stability of E_4.

CONCLUSION

The discrete-time model in system (3) represents the interaction between two populations in a host–commensal relationship. Both populations grow logistically according to their respective environmental carrying capacities, while harvesting is applied to the commensal population. Based on the analysis, when Δt is sufficiently small, the system stability remains consistent between the 4th-order Runge–Kutta scheme and the NFSD scheme. However, as Δt increases, this consistency is no longer maintained. The solution obtained using the NFSD scheme continues to converge toward the equilibrium points $E_3$and E_4, whereas the solution from the 4th-order Runge–Kutta scheme diverges from these points. Therefore, it can be concluded that the NFSD scheme is more flexible and numerically stable, as it preserves consistent results for both small and large time-step values. To support the dynamic analysis results, then the numerical simulation are performed.

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