Jurnal Matematika Murni dan Terapan "εpsilon"
Vol.9 No.2 (2015) Hal. 1- 7

NUMERICAL SOLUTION OF ABSORBING BOUNDARY CONDITION IN PADÉ APPROXIMATION

Mohammad Mahfuzh Shiddiq
Program Studi Matematika Fakultas MIPA
Universitas Lambung Mangkurat
E-mail: mmahfuzhs@unlam.ac.id

ABSTRACT

Most of numerical calculation methods of partial differential equations which have an infinite natural domain require an artificial boundary condition for reducing the size of its natural domain. One of these boundary conditions is absorbing boundary condition. This paper will construct an absorbing boundary condition with Padé approximations and numerical solution by finding the difference equation.

Keywords: Absorbing Boundary Condition, Padé Approximation, Acoustic Wave Equation

1. INTRODUCTION

Since 1970's many authors who researched on wave propagation model interested on finding model that can reduce unwanted reflection from the edges of the model (see [1, 2, 3, 4, 5]). This is motivated by avoidance a huge amount calculation when solving wave equation which is solved in infinite media on real world. Hence, it is necessary to consider a finite sub domain and use artificial boundary in boundary condition in such a way that solution in which the finite sub domain approximate the original solution. One of these boundary conditions called absorbing boundary condition that is when the energy in the artificial domain decreases with respect to the time.

Shiddiq [5] find absorbing boundary conditions for two dimensional acoustics wave equation that can minimize reflected wave at the bound and numerical solution with difference equation solved in Shiddiq [6]. On other hand, there are many methods to solve this wave equation, e.g. see [7] and [8]. In this paper, we find solution two dimensional acoustics wave equation that Shiddiq [5] worked in approximation way. We use Padé approximant to approach boundary conditions which defined in wave propagation problem.

2. PRELIMINARIES

2.1. Acoustic wave propagation problem

Acoustic waves are a type of longitudinal waves that propagate by means of adiabatic compression and decompression. Acoustic waves travel with the speed of sound which depends on the medium they're passing through. It exhibit phenomena like diffraction, reflection and interference.

Let two dimensional acoustics wave propagation problem given by

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0 \qquad (1)$$

defined at medium bounded by the form

$$\widetilde{D}=\left\{(x,y,t)\middle|-L\le x\le L, 0\le y\le M, 0\le t\le T\right\}$$

and $c=\omega/k$ denotes the velocity of the medium, ω is angular frequency and k is wave number . Wave equation (1) equipped with initial condition

$$u(x,y,0)=F(x),\ \frac{\partial u}{\partial t}(x,y,0)=0 \qquad (2)$$

Shiddiq in [5] told that (1) and (2) together with Dirichlet boundary conditions, that is

$$u(\pm L,y,t)=0,\ u(x,M,t)=0 \qquad (3)$$

or Neumann boundary conditions, that is

$$\frac{\partial u(\pm L,y,t)}{\partial x}=0,\ \frac{\partial u(x,M,t)}{\partial y}=0 \qquad (4)$$

produce reflected wave whose same amplitude with incidence wave at bound i.e. reflection coefficient is 1. This value of reflection coefficient obtained from substituting solution of (1) that is

$$u=e^{i(\omega t-kx\cos\theta\pm ky\sin\theta)}+Re^{i(\omega t+kx\cos\theta\pm ky\sin\theta)} \qquad (5)$$

where θ is angle between plane wave and x axis and R is reflection coefficient.

2.2. Pade approximant

Padé approximant is that rational function (of a specified order) whose power series expansion agrees with a given power series to highest possible order. If the rational function is

$$R_{(M,N)}(x)=\frac{\sum_{k=0}^{M}a^k x^k}{1+\sum_{k=1}^{N}b^k x^k}$$

then $R_{(M,N)}(x)$ is said to be a Padé approximant of order (M,N) to the series

$$f(x)=\sum_{k=0}^{\infty}c^k x^k$$

For example padé approximation for square root is given by formula

$$\sqrt{1+x} \approx 1+\sum_{j=1}^{m} \frac{a_j^{(m)} x}{1+b_j^{(m)} x} \tag{6}$$

where

$$a_j^{(m)} = \frac{2}{2m+1}\sin^2 \frac{j\pi}{2m+1};\; a_j^{(m)} = \cos^2 \frac{j\pi}{2m+1};$$

(Baker and Morris [9])

2.3. Numerical solution of wave equation

One of approximation that use to approximate differential equation in numerical solution is Taylor series. Let $f(x)$ is one variable function represent in Taylor series

$$f(x) = f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)(\Delta x)^2}{2!} + \frac{f'''(x_0)(\Delta x)^3}{3!} + \cdots + \frac{f^{(n)}(x_0)(\Delta x)^n}{n!} + \cdots \tag{7}$$

with $\Delta x = x - x_0$. If value of function f about x_0 is given, that is $f(x_0)$, $f(x-\Delta x)$, and $f(x+\Delta x)$ then from Taylor series we obtain

$$f'(x_0) \approx \frac{f(x+\Delta x) - f(x_0)}{\Delta x} \tag{8}$$

$$f'(x_0) \approx \frac{f(x_0) - f(x-\Delta x)}{\Delta x} \tag{9}$$

$$f'(x_0) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} \tag{10}$$

$$f''(x_0) \approx \frac{f(x+\Delta x) - 2f(x_0) + f(x-\Delta x)}{\Delta x^2} \tag{11}$$

which called forward difference, backward difference, center difference for first derivative and center difference for second derivative respectively (Morton and Mayers [10]).

Then from (11) and Mathew and Kurtis [11] follows that difference equation for wave equation (1) is given by

$$\frac{1}{c^2}\frac{u_{i,j}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1}}{k^2} = \frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{h^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{p^2} \tag{12}$$

where $\Delta x = h, \Delta y = p$ and $\Delta t = k$ for $i = 1,2,\ldots,M_1;\ j = 1,2,\ldots,M_2$ and $n = 1,2,\ldots,T_1$. If value of u at n and $n-1$ is given, then (12) became

$$u_{i,j}^{n+1} = 2u_{i,j}^{n} - u_{i,j}^{n-1} + r^2\left(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}\right) + s^2\left(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right) \qquad (13)$$

with $r = \frac{c\Delta t}{\Delta x}$ and $s = \frac{c\Delta t}{\Delta y}$.

3. MAIN RESULT

Consider acoustics wave equation in two dimensions with the form

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

Special families of solution the wave equation representing wave travelling to the right are given by the plane waves

$$u = e^{i(\omega t - kx\cos\theta + ky\sin\theta)}, \quad 0 \le \theta \le \frac{\pi}{2} \qquad (14)$$

If (x, y) is held fixed, one first order differential boundary condition which annihilates u has the form

$$\left(\frac{\partial}{\partial x} + ik\cos\theta\right)u\bigg|_{x=L} = 0 \qquad (15)$$

Then all waves in (14) with boundary condition (15) produce no reflection at $x = L$. However this perfectly absorbing boundary condition is necessarily nonlocal in both space and time. Thus it needed local approximation to absorbing boundary (15) that satisfied two additional requirements, that is local boundary condition and lead well-posed boundary value problem for the wave equation.

Consider again the symbol of boundary condition (15) and we know that

$$\frac{\partial}{\partial x} + ik\cos\theta = \frac{\partial}{\partial x} + ik\sqrt{1-\sin^2\theta} \qquad (16)$$

We know that (16) in the form square root. So now we would like approximate boundary condition (16) with Padé approximant.

We use first order padé approximant for square root, that is $\sqrt{1-x} = 1 + O\left(|x|^2\right)$ we obtain

$$\frac{\partial}{\partial x} + ik\sqrt{1-\sin^2\theta} = \frac{\partial}{\partial x} + ik$$

and remembering that $ik = \frac{1}{c}i\omega$ corresponds to $\frac{1}{c}\frac{\partial}{\partial t}$ so we get

First approximation absorbing boundary condition

$$\left[\frac{\partial}{\partial x}+\frac{1}{c}\frac{\partial}{\partial t}\right]u=0 \tag{17}$$

Then we investigate value of reflection coefficient by substituting u defined in (5), that is

$$\left[\frac{\partial}{\partial x}+\frac{1}{c}\frac{\partial}{\partial t}\right]u=\frac{\partial}{\partial x}\left(e^{i(\omega t-kx\cos\theta\pm ky\sin\theta)}+Re^{i(\omega t+kx\cos\theta\pm ky\sin\theta)}\right)$$

$$+\frac{1}{c}\frac{\partial}{\partial t}\left(e^{i(\omega t-kx\cos\theta\pm ky\sin\theta)}+Re^{i(\omega t+kx\cos\theta\pm ky\sin\theta)}\right)$$

$$0 \qquad =ik\cos\theta\left(-e^{i(\omega t-kx\cos\theta\pm ky\sin\theta)}+Re^{i(\omega t+kx\cos\theta\pm ky\sin\theta)}\right)+$$

$$\frac{i\omega}{c}\left(e^{i(\omega t-kx\cos\theta\pm ky\sin\theta)}+Re^{i(\omega t+kx\cos\theta\pm ky\sin\theta)}\right)$$

Multiplying by $1/ik$, we get

$$|R|=\frac{1-\cos\theta}{1+\cos\theta}=\widetilde{R}_1(\theta)$$

Secondly, we using second Padé approximant for square root, that is

$$\sqrt{1-x}=\frac{4+3x}{4+x}+O\left(|x|^3\right)$$

Using similar way as in the above process, we obtain

Second approximation absorbing boundary condition

$$\left[\frac{1}{c^2}\frac{\partial^3}{\partial x\partial t^2}-\frac{1}{4}\frac{\partial^3}{\partial x\partial y^2}+\frac{1}{c^3}\frac{\partial^3}{\partial t^3}-\frac{3}{4c}\frac{\partial^3}{\partial y^2\partial t}\right]u=0$$

and

$$|R|=\frac{\cos\theta-\frac{1}{4}\cos\theta\sin^2\theta-1+\frac{3}{4}\sin^2\theta}{\cos\theta-\frac{1}{4}\cos\theta\sin^2\theta+1-\frac{3}{4}\sin^2\theta}=\widetilde{R}_2(\theta)$$

Using third padé approximant for square root, that is

$$\sqrt{1-x}=\frac{16-20x+5x^2}{16-12x+x^2}+O\left(|x|^4\right)$$

Then we get

Third approximation absorbing boundary condition

$$\left[-\frac{1}{c^4}\frac{\partial^5}{\partial t^4\partial x}-\frac{3}{4}\frac{1}{c^2}\frac{\partial^5}{\partial t^2\partial x\partial y^2}-\frac{1}{16}\frac{\partial^5}{\partial y^5}-\frac{1}{c^5}\frac{\partial^5}{\partial t^5}-\frac{1}{c^3}\frac{\partial^5}{\partial t^3\partial y^2}-\frac{1}{c}\frac{\partial^5}{\partial t\partial y^4}\right]u=0$$

and

$$|R|=\frac{\cos\theta+\frac{3}{4}\cos\theta\sin^2\theta-\frac{1}{16}\sin^5\theta-1-\sin^2\theta-\sin^4\theta}{-\cos\theta-\frac{3}{4}\cos\theta\sin^2\theta-\frac{1}{16}\sin^5\theta-1-\sin^2\theta-\sin^4\theta}=\widetilde{R}_3(\theta)$$

Value of reflection coefficient for three boundary conditions are given in Table 1 below.

Table 1. Value of Reflection Coefficient

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\widetilde{R}_1(\theta)$	0	0.0718	0.1716	0.3333	1
$\widetilde{R}_2(\theta)$	0	0.0004	0.0051	0.0370	1
$\widetilde{R}_3(\theta)$	0	0.1221	0.2886	0.4999	1

From Table 1 we conclude that second approximation for absorbing boundary is the best because it produces the smallest amplitude of reflected wave.

Next step, we would like approximate (17) with Taylor expansion and finite difference. Before that, we remembering expansion of two variable function about the point (a,b) given by

$$f(x,y)=f(a,b)+\frac{\partial f(a,b)}{\partial x}(x-a)+\frac{\partial f(a,b)}{\partial y}(y-b)+R_1(x,y) \tag{18}$$

where $R_1(x,y)$ is remainder of Taylor polynomial degree 1. Therefore according (18) then form (17) can be written

$$\frac{\partial u(L,y,t)}{\partial x}+\frac{1}{c}\frac{\partial u(L,y,t)}{\partial t}=\frac{u^n_{M_1,j}-u^n_{M_1-1,j}}{\Delta x}+\frac{1}{c}\left(\frac{u^{n+1}_{M_1,j}-u^n_{M_1,j}}{\Delta t}\right)$$

$$0\quad =c\Delta t\left(u^n_{M_1,j}-u^n_{M_1-1,j}\right)+\Delta x\left(u^{n+1}_{M_1,j}-u^n_{M_1,j}\right)$$

So we get

$$u^{n+1}_{M_1,j}\quad =\frac{c\Delta t}{\Delta x}\left(u^n_{M_1-1,j}-u^n_{M_1,j}\right)+\Delta x\, u^n_{M_1,j} \tag{19}$$

Therefore, equation (19) is finite difference equation for (17).

4. CONCLUSION

In this paper we have approximate boundary condition with Padé approximant. The best approximation is occurred while it uses second order Padé approximant. It can be seen at table 1 that second approximation produces the smallest amplitude of reflected wave in all angle.

REFERENCES

[1] Lindmann, E.L. 1975. "Free Space" boundary conditions for the time domain dependent wave equation. *J.Comput.Phys.* **18**. 66–78.
[2] Smith, W.D. 1974. A non reflecting plane boundary for wave propagartion problems. *J.Comput.Phys.* **15**. 492–503
[3] Joly P. and Tuomela J. 1997. A new theoretical approach to absorbing layer. *Siam J. Numer. Anal.* **34**(2). 671–698
[4] Feven T and Jiang Hong. 1999. Absorbing boundary conditions for the schrödinger equation. *Siam J. Sci. Comput.* **21** (1). 255–282
[5] Shiddiq, M. Mahfuzh. 2011. Syarat batas serap pada gelombang akustik dua dimensi. *Jurnal Matematika Murni dan Terapan Epsilon.* **05** (2)
[6] Shiddiq, M. Mahfuzh. 2013. Numerical solution of absorbing boundary conditions on two dimensional acoustic wave. *Jurnal Matematika Murni dan Terapan Epsilon.* **07** (1)
[7] Higdon R. 1987. Numerical absorbing boundary conditions for wave equation. *Math. Comp.* **49**. 65–90
[8] Rowley C and Colonius T. 2000. Discretely nonreflecting boundary condition for linear hyperbolic system. *J. Comput. Phys.* **15**. 500–538
[9] Baker G.A and Morris Grave P. 1996. *Padé approximation*, 2nd edition. Cambridge University Press.
[10] Morton K.W, and Mayers D.F. 2005. *Numerical solution of partial differential equation, An Introduction*. Cambridge University Press
[11] Mathew J.H and Kurtis F. 1999. *Numerical Method Using Matlab; Third edition*. Prentice-Hall Inc. New York