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NUMERICAL SOLUTION OF ABSORBING BOUNDARY CONDITIONS ON TWO DIMENSIONAL ACOUSTIC WAVE

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Abstrak. Pada makalah ini akan ditentukan solusi numerik dari persamaan gelombang akustik dua dimensi dengan syarat batas serap yang diperoleh pada Shiddiq [9]. Metode yang dipakai adalah menentukan persamaan beda hingga pada masalah perambatan gelombang akustik dua dimensi yang memenuhi syarat batas serap.

Kata Kunci : Persamaan gelombang dua dimensi, Syarat batas serap, Persamaan beda hingga.

Abstract. In this paper will be determined numerical solution of two-dimensional acoustic wave equation with absorbing boundary conditions that obtained at Siddiq [9]. Method used is determine finite difference equation in two-dimensional acoustic wave propagation problems satisfied absorbing boundary conditions.

Keywords : Two dimensional wave equation, Absorbing boundary conditions, Finite difference equation.

1. Introduction

Wave equation problems on real world solved in infinite media. This problem can not solved in finite media. So it need a model which could solved in finite media. In other hand, limited computer calculation on approximation solution of difference equation also need boundaries to get a finite model. Hence to avoid infinite calculation or a huge amount calculation when solving numerically a wave equation, it is often introduces artificial boundaries so that the problem one gets is well-posed, and its solution is "as close as possible" to exact solution.

Several author was studied in many cases, for instance in geophysics; see [1, 3, 7, 8, 9]. Shiddiq [9] mentioned that in two dimensional acoustic wave together with Neumann boundary problem and Dirichlet boundary condition produce reflected wave whose big amplitudo at bound. So the problem is find boundary conditions which can eliminated reflected wave at bound. This boundary condition called absorbing boundary condition. In Shiddiq [9] find five absorbing boundary condition that can minimize reflected wave at bound. In this paper we determine difference equation for absorbing boundary condition that Shiddiq [9] find.

2. Preliminaries

Wave equation is a hyperbolic partial differential equation. It concern a time variable t, spatial variables $x_1, x_2, \ldots, x_n$ and scalar function $u(x_1, x_2, \ldots, x_n; t)$. Feynman [2] provided a derivation of the acoustic wave equation that describe the behaviour of vector field velocity is given

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

where u is function $u(x_1, x_2, \ldots, x_n; t)$, ∇^2 is the Laplace operator, and c is speed of wave propagation.

If wave equation (1) together with one or more condition that called boundary condition then it called a *boundary value problem*. A solution of boundary value problem is solution to the wave equation which also satisfy boundary conditions. A large class of important boundary value problems are the *Sturm Liouville problems*. Among the earliest boundary value problems to be studied is the Dirichlet problem. If the boundary gives a value to the normal derivative of the problem then it is a *Neumann boundary condition*. If the boundary gives a value to the problem then it is a *Dirichlet boundary condition*. If the boundary has the form of a curve or surface that gives a value to the normal derivative and the variable itself then it is a *Cauchy boundary condition* (Polyanin [6]).

Solution of partial differential equation can be achieved with numerical solution. There are many method of numerical solution, one of them is finite difference method. In this method, functions are represented by their values at certain grid points and derivatives are approximated through differences in these values. First, assuming the function whose derivatives are to be approximated is properly-behaved, by Taylor's theorem, we can create a Taylor Series expansion

$$f(x) = f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)}{2!}(\Delta x)^2 + \frac{f'''(x_0)}{3!}(\Delta x)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(\Delta x)^n + \cdots \tag{2}$$

with $\Delta x = x - x_0$. Let value of function $f(x_0), f(x - \Delta x), f(x + \Delta x)$ is given then

Taylor series can write

$$\begin{aligned} f(x-\Delta x) &= f(x_0) - f'(x_0)\Delta x + \frac{f''(x_0)}{2!}(\Delta x)^2 + \cdots + (-1)^n \frac{f^{(n)}(x_0)}{n!}(\Delta x)^n \\ &\quad + O(\Delta x)^{n+1} \\ f(x+\Delta x) &= f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)}{2!}(\Delta x)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(\Delta x)^n \\ &\quad + O(\Delta x)^{n+1} \end{aligned}$$

where $O(\Delta x)^{n+1}$ stand for accuracy order. Hence derivetives of function have the form

$$f'(x_0) \approx \frac{f(x+\Delta x) - f(x_0)}{\Delta x} \tag{3}$$

$$f'(x_0) \approx \frac{f(x_0) - f(x-\Delta x)}{\Delta x} \tag{4}$$

$$f'(x_0) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} \tag{5}$$

$$f''(x_0) \approx \frac{f(x+\Delta x) - 2f(x_0) + f(x-\Delta x)}{\Delta x^2} \tag{6}$$

Those derivatives approximation form (3) to (6) called (*forward difference*), *backward difference*), (*center difference*) for first derivative and (*center difference*) for second derivative (Morton [5]).

3. Main Result

Consider two dimensional acoustic wave equation propagation problem :

$$\frac{1}{c^2}\frac{\partial u}{\partial t^2} = \frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2} \tag{7}$$

for $-L \leq x \leq L, 0 \leq y \leq M, 0 \leq t \leq T$ with initial condition

$$u(x,y,0) = F(x), \qquad \frac{\partial u}{\partial t}(x,y,0) = 0 \tag{8}$$

and absorbing boundary conditions

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \cdot \left(\frac{p}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u = 0, \quad x = L \tag{9}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) \cdot \left(\frac{p}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0, \quad x = -L \tag{10}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right) \cdot \left(\frac{p}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right) u = 0, \quad y = M \tag{11}$$

The mathematical problem then is to determine solution of wave equation (7) that also satisfies initial conditions (8) and boundary conditions (9) – (11). In this case we determine numerical solution of problems which find solution with difference equation of problems. One way to numerically solve this equation is to approximate all the derivatives by finite differences. So, according (6) and Mattew and Kurtis [4] follow that difference equation for wave equation (7) is

$$\frac{1}{c^2}\frac{u_{i,j}^{n+1}-2u_{i,j}^{n}+u_{i,j}^{n-1}}{k^2}=\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\frac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{p^2} \tag{12}$$

with $\Delta x = h, \Delta y = p$, dan $\Delta t = k$ for $i = 1, 2, \ldots, M_1$; $j = 1, 2, \ldots, M_2$; $n = 1, 2, \ldots, T_1$. If value of u at n and $n-1$ is given, then form (12) became

$$u_{i,j}^{n+1}=2u_{i,j}^{n}-u_{i,j}^{n-1}+r^2\left(u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}\right)+s^2\left(u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}\right) \tag{13}$$

with $r = \frac{c\Delta t}{\Delta x}$ dan $s = \frac{c\Delta t}{\Delta y}$.

Now we will determine numerical approach for absorbing boundary condition (9) - (11). Consider boundary condition (9), it easy to show that boundary condition (9) equivalence with

$$\frac{1}{c}\frac{\partial^2 u}{\partial x \partial t}+\frac{\partial^2 u}{\partial x^2}+p\left(\frac{1}{c}\frac{\partial^2 u}{\partial x \partial t}+\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\right)=0, \quad x=L \tag{14}$$

then multiply (14) with Δx to get

$$\left.\frac{\Delta x}{c}\frac{\partial^2 u}{\partial x \partial t}+\frac{\Delta t}{c}\frac{\partial^2 u}{\partial t^2}+\Delta x\frac{\partial^2 u}{\partial x^2}+\Delta t\frac{\partial^2 u}{\partial x \partial t}\right]_{(L,y,t)}=0 \tag{15}$$

Before finding approximation for (15), we review first approximation two variable function $f(x, y)$ with Taylor series

$$\begin{aligned} f(x,y) &= f(a,b)+(x-a)\frac{\partial f(a,b)}{\partial x}+(y-b)\frac{\partial f(a,b)}{\partial y}+\frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f(a,b)}{\partial x^2}\right. \\ &\quad \left. +2(x-a)(y-b)\frac{\partial^2 f(a,b)}{\partial x \partial y}+(y-b)^2\frac{\partial^2 f(a,b)}{\partial y^2}\right]+R_2(x,y) \end{aligned}$$

where $R_2(x, y)$ is remainder of Taylor polynomial degree 2. Therefore according above Taylor series form (15) can be write

$$\frac{1}{c}\left(\frac{\partial u}{\partial t}(L,y,t)-\frac{\partial u}{\partial t}(L-\Delta x,y,t-\Delta t)\right)=\frac{1}{c}\left(\Delta x\frac{\partial^2 u}{\partial x \partial t}(L,y,t)+\Delta t\frac{\partial^2 u}{\partial t^2}(L,y,t)\right)$$

and

$$\frac{\partial u}{\partial x}(L,y,t)-\frac{\partial u}{\partial x}(L-\Delta x,y,t-\Delta t)=\Delta x\frac{\partial^2 u}{\partial x^2}(L,y,t)+\Delta t\frac{\partial^2 u}{\partial x \partial t}(L,y,t)$$

Adding two equation we obtain other form for (15)

$$\frac{1}{c}\frac{\partial u}{\partial t}(L,y,t)+\frac{\partial u}{\partial x}(L,y,t)-\frac{1}{c}\frac{\partial u}{\partial t}(L-\Delta x,y,t-\Delta t)-\frac{\partial u}{\partial x}(L-\Delta x,y,t-\Delta t)=0 \quad (16)$$

Next step is find difference equation for (16). If we see each term in (16), it follow that

$$\begin{aligned}
\frac{1}{c}\frac{\partial u(L,y,t)}{\partial t} &= \frac{1}{c}\left(\frac{u^{n+1}_{M_1,j}-u^{n}_{M_1,j}}{\Delta t}\right) && (17)\\
\frac{\partial u(L,y,t)}{\partial x} &= \frac{u^{n}_{M_1,j}-u^{n}_{M_1-1,j}}{\Delta x} && (18)\\
-\frac{1}{c}\frac{\partial u(L-\Delta x,y,t-\Delta t)}{\partial t} &= -\frac{1}{c}\left(\frac{u^{n}_{M_1-1,j}-u^{n-1}_{M_1-1,j}}{\Delta t}\right) && (19)\\
-\frac{\partial u(L-\Delta x,y,t-\Delta t)}{\partial x} &= -\left(\frac{u^{n-1}_{M_1-1,j}-u^{n-1}_{M_1-2,j}}{\Delta x}\right) && (20)
\end{aligned}$$

so from (17) - (20) we get

$$u^{n+1}_{M_1,j}=u^{n}_{M_1,j}+u^{n}_{M_1-1,j}-u^{n-1}_{M_1-1,j}+\frac{c\Delta t}{\Delta x}\left(u^{n}_{M_1-1,j}-u^{n}_{M_1,j}-\left[u^{n-1}_{M_1-1,j}-u^{n-1}_{M_1-2,j}\right]\right) \quad (21)$$

Therefore (21) is difference equation for boundary condition (9).

Then finding difference equation for boundary condition (10). First, multiply (10) with Δx

$$-\frac{\Delta x}{c}\frac{\partial^2 u}{\partial x\partial t}+\frac{\Delta t}{c}\frac{\partial^2 u}{\partial t^2}+\Delta x\frac{\partial^2 u}{\partial x^2}-\Delta t\frac{\partial^2 u}{\partial x\partial t}\Big]_{(-L,y,t)}=0 \quad (22)$$

Taylor series approximation for two variable function follow that (22) can be written as

$$\begin{aligned}
0 &= \frac{1}{c}\frac{\partial u}{\partial t}(-L,y,t)-\frac{\partial u}{\partial x}(-L,y,t)-\frac{1}{c}\frac{\partial u}{\partial t}(-L+\Delta x,y,t-\Delta t)\\
&\quad +\frac{\partial u}{\partial x}(-L+\Delta x,y,t-\Delta t) \qquad (23)
\end{aligned}$$

Difference equation for (23) is

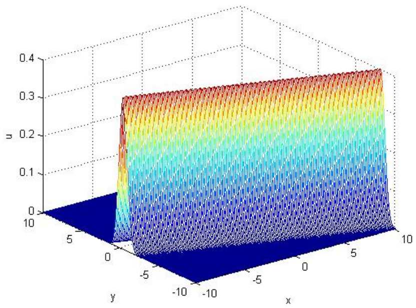
$$u^{n+1}_{1,j}=u^{n}_{1,j}+u^{n}_{2,j}-u^{n-1}_{2,j}-\frac{c\Delta t}{\Delta x}\left(u^{n}_{1,j}-u^{n}_{2,j}-\left[u^{n-1}_{3,j}-u^{n-1}_{2,j}\right]\right) \quad (24)$$

Therefore (24) is difference equation for (10).

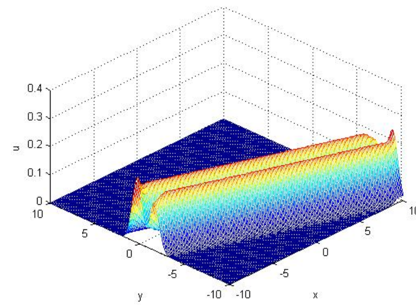
Using similar way, it follow that

$$u^{n+1}_{i,M_2}=u^{n}_{i,M_2}+u^{n}_{i,M_2-1}-u^{n-1}_{i,M_2-1}+\frac{c\Delta t}{\Delta x}\left(u^{n}_{i,M_2-1}-u^{n}_{i,M_2}-\left[u^{n-1}_{i,M_2-1}-u^{n-1}_{i,M_2-2}\right]\right) \quad (25)$$

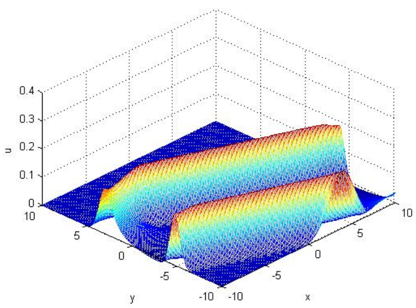
is difference equation for (11). Numerical calculation in computing program generates simulation of solution of two-dimensional acoustic wave equation with absorbing boundary conditions as in figure (1).



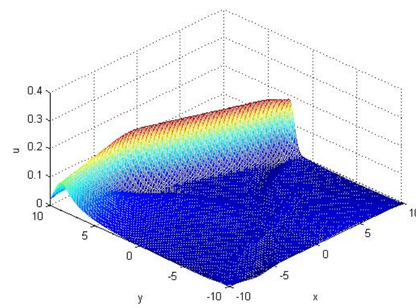
(a) outset wave



(b) cracked wave



(c) wave at bound



(d) wave at bound

Gambar 1. Absorbing Boundary Condition on 2D wave

4. Conclusion

We have solved a acoustic wave propagation problem with absorbing boundary conditions using finite difference method. But from figure (1), it can be seen that waveform does not look as usual shape that is there are something wrong at their sides. Furthermore, by analyze errors in simulation or by improving the accuracy of Taylor series or with other approximation in order to minimize error in waveform of simulation.

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